BIRKHOFF-JAMES ORTHOGONALITY IN THE GEOMETRY OF BANACH SPACES AND SOME RELATED TOPICS

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Abstract

This dissertation explores some conceptual aspects of Birkhoff-James orthogonality and exhibits its applicability in various facets of the geometry of Banach spaces. From the theoretical point of view, the local symmetry of Birkhoff-James orthogonality is studied for real Banach spaces. A characterization of symmetric points in a Banach space is obtained by employing the idea of the left-symmetric and the right-symmetric points. To illustrate the utility of the investigation, a certain class of polyhedral spaces is considered. In complex Banach spaces, a weaker notion of orthogonality, known as directional orthogonality, is explored. The said investigation provides a better understanding of smooth and non-smooth points in a Banach space. Aside from studying orthogonality in the ground space, the space of all vector-valued continuous functions on a compact topological space is considered. This comparatively weak setting facilitates the perception of the operator orthogonality in a much wider scope in the finite-dimensional context. Apart from producing some new results, several existing results are strengthened.

From the perspective of applications, smoothness induced by the numerical radius is studied in the spaces of operators. Applying the Birkhoff-James orthogonality techniques the smooth operators with respect to the numerical radius norm (nu-smooth operators) are identified in both finite and infinite-dimensional contexts. Distinction between smoothness and nu-smoothness is illustrated through supportive examples. In the other direction, the concept of level numbers and level vectors of a bounded linear operator is introduced as a generalization of the singular values and singular vectors. The distinctions between level vectors and singular vectors are discussed in relevant scenarios. The points, where an operator preserves Birkhoff-James orthogonality, is interpreted through a nice equation involving semi-inner-products. Then using Birkhoff-James orthogonality and operator norm attainment, a new approach towards singular value decomposition of a matrix is furnished.

Keywords: *Banach space, Birkhoff-James orthogonality, left-symmetric points, right-symmetric points, symmetric points, support functional, directional orthogonality, smooth points, norm attainment, numerical range, numerical radius, Toeplitz-Hausdorff Theorem, operator orthogonality, Bhatia-Šemrl Theorem, continuous vector-valued functions, bilinear forms, sesquilinear forms, nu-smoothness, level numbers, level vectors, singular value decomposition.*